A measure of the multiverse

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Abstract
I review recent progress in defining a probability measure in the inflationary
multiverse. General requirements for a satisfactory measure are formulated
and recent proposals for the measure are clarified and discussed.

PACS numbers: 98.80.Bp, 98.80.Cq

1. Introduction

String theory appears to have a multitude of solutions describing vacua with different values
of the low-energy constants. The number of vacua in this vast 'landscape' of possibilities can
be as large as 10^{500} [1–3]. In the cosmological context, high-energy vacua drive exponential
inflationary expansion of the universe. Transitions between different vacua occur through
tunnelling and quantum diffusion, with regions of different vacua nucleating and expanding in
the never-ending process of eternal inflation [4, 5]. As a result, the entire landscape of vacua
is explored.

   If indeed this kind of picture describes our universe, we will never be able to calculate
all constants of Nature from first principles. At best we may only be able to make statistical
predictions. The key problem is then to calculate the probability distribution for the constants.
It is often referred to as *the measure problem*.

   The probability P_j of observing vacuum j can be expressed as a product

$$P_j = P_j^{(\text{prior})} f_j, \tag{1}$$

where the prior probability $P_j^{(\text{prior})}$ is determined by the geography of the landscape and by
the dynamics of eternal inflation, and the selection factor f_j characterizes the chances for an
observer to evolve in vacuum j. The distribution (1) gives the probability for a randomly
picked observer to be in a given vacuum.

   It seems natural to identify the prior probability with the fraction of volume $P_j^{(V)}$ occupied
by a given vacuum and the selection factor with the number of observers $n_j^{(\text{obs})}$ per unit volume[1]
[6],

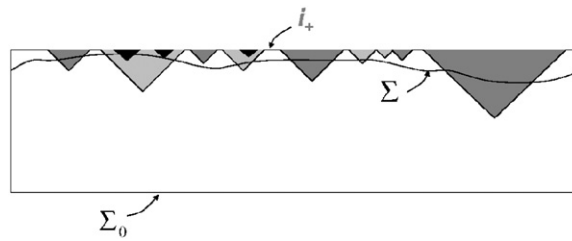[1] The product in (1) should of course be properly normalized.

Figure 1. A schematic conformal diagram for a comoving region in an eternally inflating universe. Bubbles of different vacua are represented by different shades of grey. The upper boundary of the diagram i_+ is the future timelike infinity. A surface of constant global time Σ cuts through the entire region and intersects many bubbles.

$$P_j^{(\text{prior})} \propto P_j^{(V)}, \tag{2}$$

$$f_j \propto n_j^{(\text{obs})}. \tag{3}$$

This approach, however, encounters a severe difficulty: the result sensitively depends on the choice of a spacelike hypersurface (a constant-time surface) on which the distribution is to be evaluated. This problem was uncovered by Andrei Linde and his collaborators when they first attempted to calculate volume distributions [7–9]. It eluded resolution for more than a decade, but recently there have been some promising developments, and I believe we are getting close to completely solving the problem. Here, I will briefly discuss the nature of the difficulty and then review the new proposals for P_j. Most of this discussion is based on my work with Jaume Garriga, Delia Schwartz-Perlov, Vitaly Vanchurin and Serge Winitzki [10, 11] (see also [12]).

2. Problem with global-time measure

The spacetime structure of an eternally inflating universe is schematically illustrated in figure 1. For simplicity, we shall focus on the case where transitions between different vacua occur only through bubble nucleation. The bubbles expand rapidly approaching the speed of light, so their worldsheets are well approximated by light cones. Disregarding quantum fluctuations, bubble interiors are open FRW universes [13]; they are often called 'pocket universes'. If the vacuum inside a bubble has positive energy density, it becomes a site of further bubble nucleation; we call such vacua 'recyclable'. Negative-energy vacua, on the other hand, quickly develop curvature singularities; we shall call them 'terminal vacua'.

The diagram represents a comoving region, which is initially comparable to the horizon. The initial moment is a spacelike hypersurface Σ_0, represented by the lower horizontal boundary of the diagram, while the upper boundary represents future infinity, when the region and all the bubbles become infinitely large. How can we find the fraction of volume occupied by different vacua? A natural thing to do is to consider a spacelike hypersurface Σ, which cuts through the entire region, as shown in the figure. If t is a globally defined time coordinate, then all surfaces $t = \text{const}$ will have this property. One can use, for example, the proper time along the 'comoving' geodesics orthogonal to the surface Σ_0.[2] Alternatively, one could use the so-called scale factor time, defined as a logarithm of the expansion factor along the comoving geodesics, or any other suitable time coordinate. Once the time coordinate is specified, one

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[2] The term 'comoving' is used very loosely here, since the vacuum does not define any rest frame. Any congruence of geodesics orthogonal to a smooth spacelike surface Σ_0 can be regarded as 'comoving'.

can find the fraction of volume occupied by different vacua on the surface $t = $ const and then take the limit $t \to \infty$.

Unfortunately, as I have already mentioned, the result of this calculation is sensitively dependent on one's choice of the time coordinate [7]. The reason is that the volume of an eternally inflating universe grows exponentially with time. The volumes of regions filled with all possible vacua grow exponentially as well. At any time, a substantial part of the total volume is in new bubbles which have just nucleated. Which of these bubbles are cut by the surface depends on how the surface is drawn; hence the gauge dependence of the result. Since time is an arbitrary label in General Relativity, none of the possible choices of the global time coordinate appears to be preferred. For more discussion of this gauge-dependence problem, see [14–16].

3. General requirements

At this point, it will be useful to formulate some general requirements that any satisfactory definition of P_j should comply with [10].

First, we require that P_j should not depend on gauge, that is, on arbitrary choice of a hypersurface. More generally, it should not depend on any arbitrary choices.

Second, we require that P_j should be independent of the initial conditions at the onset of inflation. The dynamics of eternal inflation is an attractor; its asymptotic behaviour has no memory of the initial state[3]. We believe that the probabilities should also have this property. Note that this condition is not satisfied by an earlier proposal in [17] and by a more recent proposal in [18].

4. A pocket-based measure

4.1. Bubble abundance p_j

We shall now discuss the new proposal for P_j, introduced in [10]. The presentation here is somewhat different from [10], but the essence is the same.

The idea is that instead of trying to compare volumes occupied by different vacua, we compare the numbers of different types of bubbles (pocket universes). Thus, instead of equation (2), we make the assignment

$$P_j^{(\text{prior})} \propto p_j, \tag{4}$$

where p_j is the abundance of j-type bubbles. (We shall see later that the volume expansion in this approach is accounted for in the selection factor f_j.)

The definition of p_j is a tricky business, because the total number of bubbles is infinite, even in a region of a finite comoving size[4]. We thus need to introduce some sort of a cutoff.

The proposal of [10] is very simple: count only bubbles greater than a certain comoving size ϵ, and then take the limit $\epsilon \to 0$. That is,

$$p_j = \lim_{\epsilon \to 0} \frac{N_j(>\epsilon)}{N(>\epsilon)}. \tag{5}$$

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[3] I assume that any vacuum is accessible through bubble nucleation from any other vacuum. Alternatively, if the landscape splits into several disconnected domains which cannot be accessed from one another, each domain will be characterized by an independent probability distribution.

[4] The problem of calculating p_j is somewhat similar to the question of what fraction of all natural numbers are odd. The answer depends on how the numbers are ordered. With the standard ordering, $1, 2, 3, 4, \ldots$, the fraction of odd numbers in a long stretch of the sequence is $1/2$, but if one uses an alternative ordering $1, 3, 2, 5, 7, 4, \ldots$, the result would be $2/3$. One could argue that, in the case of integers, the standard ordering is more natural, so the correct answer is $1/2$. Here we seek an analogous ordering criterion for the bubbles.

To define the comoving size, one has to specify a congruence of 'comoving' geodesics emanating (orthogonally) from some initial spacelike hypersurface Σ_0. As they extend to the future, the geodesics will generally cross a number of bubbles before ending up in one of the terminal bubbles, where inflation comes to an end. There will also be a (measure zero) set of geodesics which never hit terminal bubbles. The starting points of these geodesics on Σ_0 provide a mapping of the eternally inflating fractal [19–21], consisting of points on i_+ where inflation never ends. In the same manner, each bubble encountered by the geodesics will also be mapped on Σ_0, and we can define the comoving size of a bubble as the volume of its image on Σ_0. (The volume of a bubble is calculated including all the daughter bubbles that nucleate within it.) Throughout this paper, we disregard bubble collisions.

I will now argue that the above prescription satisfies the requirements formulated in section 3.

In an inflating spacetime, geodesics rapidly diverge, so bubbles formed at later times have a smaller comoving size. (The comoving size of a bubble is set by the horizon at the time of bubble nucleation.) The bubble counting can be done in an arbitrarily small neighbourhood δ of any point belonging to the 'eternal fractal' image on Σ_0. Every such neighbourhood (except a set of relative measure zero) will contain an infinite number of bubbles of all kinds and will be dominated by bubbles formed at very late times and having very small comoving sizes. The resulting values of p_j, obtained in the limit of bubble size $\epsilon \to 0$, will be the same in all such neighbourhoods, because of the universal asymptotic behaviour of eternal inflation. The same result will also hold in any finite-size region on Σ_0 (provided that it contains at least one 'eternal point').

The values of p_j are independent of the choice of the initial hypersurface Σ_0. Once again, this is a consequence of the universal, attractor behaviour of eternal inflation. Mathematically, this is reflected in the fact that the asymptotic distributions obtained from the Fokker–Planck equation (in the case of slow-roll models [4, 7, 22]) and from the master equation (in the case of bubble nucleation models [23]) do not depend on the initial surface that was used to define the comoving congruence[5].

The condition of orthogonality between the congruence and the hypersurface Σ_0 can be relaxed. Suppose we change Σ_0 while keeping the congruence fixed, so that the congruence and Σ_0 are no longer orthogonal. Once again, focusing on the vicinity of an eternal point, any change of the hypersurface amounts to a constant rescaling of all bubble sizes and has no effect on p_j.

Moreover, although we use the metric on Σ_0 to compare the bubble sizes, the results are unaffected by arbitrary smooth transformations of the metric. Any such transformation will locally be seen as a linear transformation, which amounts to a constant rescaling. In a sufficiently small patch of Σ_0, all bubble volumes are rescaled in the same way, so the bubble counting should not be affected.

The results obtained using this method are also independent of the initial conditions at the onset of eternal inflation. This simply follows from the facts that the bubble counting is dominated by late times and that the asymptotic behaviour in eternal inflation is independent of the initial state.

The calculation of bubble abundances, defined by equation (5), can be reduced to an eigenvalue problem for a matrix constructed out of the transition rates between different vacua

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[5] Another way to see this is to consider a small patch of Σ_0 including an eternal point. If the patch is small enough, it can be regarded as flat. Then any change of Σ_0 will amount to changing the orientation of its normal, that is, the 4-velocity of the 'comoving' congruence. But since there is no preferred frame in the inflating de Sitter-like spacetime, all choices will result in the same asymptotic behaviour and will yield identical values of p_j.

[10][6]. This prescription has been tried on some simple models and appears to give reasonable results [10, 25]. For example, if there is a single false vacuum, which can decay into a number of vacua with nucleation rates Γ_j, one finds

$$p_j \propto \Gamma_j, \tag{6}$$

as intuitively expected.

4.2. Eternal observers

Another interesting special case is that of full recycling. If all vacua have positive energy density, $\rho_j > 0$, there are no terminal vacua and all geodesics of the congruence represent 'eternal observers', who endlessly transit from one vacuum to another, exploring the entire landscape. The distribution p_j in this case can be found in a closed form [11]:

$$p_j \propto \sum_i \Gamma_{ji} \, e^{S_i}, \tag{7}$$

where Γ_{ji} is the nucleation rate of j-type bubbles in vacuum i, $S_i = \pi / H_i^2$ is the Gibbons–Hawking entropy of de Sitter space and

$$H_j = (8\pi G \rho_j / 3)^{1/2} \tag{8}$$

is the expansion rate corresponding to the local vacuum energy density ρ_j.

In the case of full recycling, the bubble abundance p_j can also be defined as the frequency at which j-type bubbles are visited along the worldline of an eternal observer. This definition involves observations accessible to a single observer—a property that some string theorists find desirable [2, 18]. It has been shown in [11] that this eternal-observer definition gives the same result (7) as the pocket-based measure of [10].

For example, in the simplest case of only two vacua, equation (7) gives $p_1 \propto \Gamma_{12} \, e^{S_2}$, $p_2 \propto \Gamma_{21} \, e^{S_1}$, and using the property [26]

$$\Gamma_{ij} / \Gamma_{ji} = e^{S_i - S_j}, \tag{9}$$

we obtain $p_1 = p_2 = 0.5$. This is, of course, in agreement with the frequency of visiting the two vacua: the frequency should be the same, since the eternal observer goes back and forth between them.

4.3. An equivalent proposal

An alternative prescription for p_j has been suggested by Easther, Lim and Martin [27]. They randomly select a large number $\mathcal{N}$ of points on a compact patch of a spacelike hypersurface Σ_0 in the inflating part of spacetime. They follow the geodesics emanating from these points and check which bubbles they cross. The bubble abundance is then defined as

$$p_j = \lim_{\mathcal{N} \to \infty} \frac{\mathcal{N}_j}{\mathcal{N}}, \tag{10}$$

where $\mathcal{N}_j$ is the number of type-j bubbles crossed by at least one geodesic.

As the number of points is increased, the average distance δ between them on Σ_0 gets smaller, so most bubbles of comoving volume larger than $\epsilon \sim \delta^3$ are counted. In the limit of

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[6] The calculation in [10] assumes that the divergence of geodesics is everywhere determined by the local vacuum energy density. This is somewhat inaccurate, since it ignores the brief transition periods following the bubble crossings and the focusing effect of the domain walls. The accuracy of the method is expected to be up to factors $O(1)$. A more detailed discussion will be given elsewhere [24].

$\mathcal{N} \to \infty$, we have $\epsilon \to 0$, and it is not difficult to see that this prescription is equivalent to that described in the preceding subsection. (For a rigorous proof, see *note added* in [10].)

Easther *et al* argue that the values of p_j in (10) are independent of the choice of measure on Σ_0. This is consistent with our analysis. They also argue that the initial velocities of the worldlines on Σ_0 can be chosen at random without affecting the p_j. I think this statement needs to be modified. If the velocities at neighbouring points are chosen independently, then in the limit $n \to \infty$ the velocity distribution on Σ_0 will be very singular, and I see no reason to expect that p_j will remain unchanged. On the other hand, we do expect p_j to be invariant under continuous variations of the geodesic congruence, as explained in section 4.1.

4.4. Bousso's proposal

Raphael Bousso [18] (see also [28]) suggested an extension of the prescription in [11] to the case when there are some terminal bubbles, so the observers are generally not eternal. The idea is to start with an ensemble of observers characterized by some initial distribution. All observers, except a set of measure zero, will end up in terminal bubbles after visiting a certain number of recyclable bubbles. Bousso's proposal is that the measure p_j should be proportional to the total number of times the observers in the ensemble visit bubbles of type j.

The resulting measure is strongly dependent on the initial distribution function, so one has to address the question of where that distribution comes from. Bousso suggests it might be derived from the wavefunction of the universe Ψ. The usual interpretation of Ψ is that it gives probabilities for different initial states as the universe nucleates out of nothing. The nucleation is followed by eternal inflation, which produces an unlimited number of all possible bubbles, so the initial state is quickly forgotten. Bousso's proposal is based on a very different, holographic view, which asserts that the region outside the horizon should be completely excluded from consideration. Hence, one deals with an ensemble of disconnected horizon-size regions nucleating out of nothing. For someone not initiated in holography, this view is very hard to adopt, but as long as it is mathematically consistent, one can work out its predictions and compare them with the data.

5. The selection factor f_j

The selection factor f_j should characterize the relative number of observers in different types of pockets. As I already mentioned, the interior spacetime of a pocket is that of an open FRW universe, so each pocket that has any observers in it has an infinite number of them. In order to compare the numbers of observers, we will have to define a comoving length scale R_j on which observers are to be counted in bubbles of type j. The first thing that comes to mind is to set R_j to be the same for all bubbles. However, this is not enough. The expansion rate is different in different bubbles, so the physical length scales corresponding to R_j will not remain equal, even if they were equal at some moment. We could specify the times t_j at which R_j are set to be equal, but any such choice would be subject to the criticism of being arbitrary.

A possible way around this difficulty was proposed in [10]. At early times after nucleation, the dynamics of open FRW universes inside bubbles is dominated by the curvature, with the scale factor given by

$$a_j(t) \approx t \tag{11}$$

for all types of bubbles. For example, for a quasi-de Sitter bubble interior,

$$a_j(t) \approx H_j^{-1} \sinh(H_j t), \tag{12}$$

where H_j is given by equation (8). The specific form of the scale factor at late times is not important for our argument. The point is that for $t \ll H_j^{-1}$ all bubble spacetimes are nearly identical, with the scale factor (11).

The proposal of [10] is that the reference scales should be chosen so that R_j are the same at some small $t = \tau$ (same for all bubbles). The choice of τ is unimportant, as long as $\tau \ll H_j^{-1}$ for all j. Then, up to a constant, the physical length corresponding to R_j is

$$R_j^{(\text{phys})}(t) = a_j(t). \tag{13}$$

For times $t \gg H_j^{-1}$, this can be expressed as

$$R_j^{(\text{phys})}(t) \approx H_j^{-1} Z_j(t), \tag{14}$$

where Z_j is the expansion factor since the onset of the inflationary expansion inside the bubble $\left(t \sim H_j^{-1}\right)$.

Alternatively, $R_j^{(\text{phys})}$ in (14) can be identified as the curvature scale. It is the characteristic large-scale curvature radius of the bubble universe. This definition makes no reference to early times close to the bubble nucleation: the curvature radius can be found at any time. It is, in principle, a measurable quantity.

The selection factor f_j can thus be written as

$$f_j \propto n_j, \tag{15}$$

where n_j is the number of observers who will evolve per unit comoving volume (normalized at the same $\tau \ll H_j^{-1}$ for all bubbles). The calculation of n_j is of course a challenging problem; I will not address it here.

It follows from equations (14) and (15) that large inflation inside bubbles is rewarded with our definition of the measure. An inflationary expansion by a factor Z enhances the probability by Z^3.

6. Continuous variables

Our prescription for the measure can be straightforwardly generalized to the case when, in addition to bubbles, there are some continuously varying fields X. Equation (4) for the prior is replaced by

$$P_j^{(\text{prior})} \propto p_j \hat{P}_j(X), \tag{16}$$

where $\hat{P}_j(X)$ is the normalized distribution for X in a bubble of type j at $t = \tau \ll H_j^{-1}$,

$$\int \hat{P}_j(X) \, \mathrm{d}X = 1. \tag{17}$$

This distribution is determined by the dynamics of quantum fields X during inflation. It can be calculated analytically or numerically, using the methods of [10, 29].

Equation (15) for the selection factor is replaced by

$$f_j(X) \propto n_j(X). \tag{18}$$

7. Discussion

The above definition of the measure is just a proposal. We have not derived it from first principles. In fact, there is no guarantee that there is some unique measure that can be used

for making predictions in the multiverse. How, then, can we ever know that we made the right choice out of all possible options?

What I find encouraging is that even a single definition of measure that satisfies some basic requirements proved very difficult to find. It is also reassuring that alternative prescriptions suggested in [27] and [11] turned out to be equivalent to the pocket-based measure of [10].

Here, we required that the measure should not depend on any arbitrary choices, such as the choice of gauge or of a spacelike hypersurface, and that it should be independent of the initial conditions at the onset of inflation. These conditions, however, do not specify the measure uniquely. For example, a flat measure, $p_j = $ const for all j, clearly satisfies the conditions. It would be interesting to formulate a set of requirements which selects a unique definition of the measure.

Bubbles of different types can generally collide with domain walls forming to separate the different vacua. Our prescription for the measure needs to be generalized to include these processes. Another necessary extension is to the case where transitions between vacua can occur through quantum diffusion. (Some steps in this direction have been made in [10].)

The ultimate test of any proposed measure will be a comparison of its predictions with observations. The first attempts in this direction have already produced some intriguing results [25, 30–36].

Acknowledgments

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